Symmetric functions of the Roots

Symmetric function :→

Definition :– If an expression involving the roots of an equation remains unaltered when any two roots are interchanged, then such an expression is called a symmetric function.

Ex :→ $\alpha^2\beta + \alpha^2\gamma + \beta^2\alpha + \beta^2\gamma + \gamma^2\alpha + \gamma^2\beta$.
is a symmetric function of the roots α, β, γ of a cubic equation.

Ex :→ Calculate the values of the following symmetric functions for the cubic equation. $x^3 + px^2 + qx + r = 0$ whose roots are α, β, γ:

(i) $\Sigma\alpha^2$

Solution :→ $\because \alpha, \beta, \gamma$ are the roots of the equation $x^3 + px^2 + qx + r = 0$, we have

$$\Sigma\alpha = \alpha + \beta + \gamma = -p$$

$$\Sigma\alpha\beta = \alpha\beta + \alpha\gamma + \beta\gamma = q$$

& $\alpha\beta\gamma = -r$.

We have,

(i) $\underline{\sum \alpha^2}$.

$$\rightarrow (\sum \alpha)^2 = (\alpha + \beta + \gamma)^2$$

$$= \alpha^2 + \beta^2 + \gamma^2 + 2\alpha\beta + 2\alpha\gamma + 2\beta\gamma$$

$$\Rightarrow (\sum \alpha)^2 = \sum \alpha^2 + 2\sum \alpha\beta$$

$$\Rightarrow \therefore \sum \alpha^2 = (\sum \alpha)^2 - 2\sum \alpha\beta$$

$$= (-p)^2 - 2q = p^2 - 2q.$$ Ans.

(ii) $\underline{\sum \alpha^2\beta^2}$.

$$\rightarrow (\sum \alpha\beta)^2 = (\alpha\beta + \beta\gamma + \gamma\alpha)^2$$

$$= (\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2) + 2\alpha\beta^2\gamma + 2\alpha^2\beta\gamma + 2\alpha\beta\gamma^2$$

$$= \sum \alpha^2\beta^2 + 2\alpha\beta\gamma(\alpha + \beta + \gamma)$$

$$\therefore \sum \alpha^2\beta^2 = (\sum \alpha\beta)^2 - 2\alpha\beta\gamma \sum \alpha$$

$$= q^2 - 2(-r)(-p) = q^2 - 2pr.$$

(iii) $\underline{\sum \alpha^2\beta}$

$$\sum \alpha \sum \alpha\beta = (\alpha + \beta + \gamma)(\alpha\beta + \beta\gamma + \gamma\alpha)$$

$$= \sum \alpha^2\beta + 3\alpha\beta\gamma$$

$\therefore \Sigma\alpha^2\beta = \Sigma\alpha\,\Sigma\alpha\beta - 3\alpha\beta\gamma$

$= (-p)q - 3(-r) = -pq + 3r = 3r - pq$

(iv) $\Sigma\alpha^3\beta$.

$\longrightarrow$ We have,

$\Sigma\alpha^3\beta = \alpha^3\beta + \alpha^3\gamma + \beta^3\alpha + \beta^3\gamma + \gamma^3\alpha + \gamma^3\beta$

Now,

$\Sigma\alpha^2\,\Sigma\alpha\beta = (\alpha^2+\beta^2+\gamma^2)(\alpha\beta+\beta\gamma+\gamma\alpha)$

$= (\alpha^3\beta + \alpha^2\beta\gamma + \alpha^3\gamma) + (\beta^3\alpha + \beta^3\gamma + \beta^2\alpha\gamma)$
$+ (\gamma^2\alpha\beta + \gamma^3\beta + \gamma^3\alpha)$

$= (\alpha^3\beta + \alpha^3\gamma + \beta^3\alpha + \beta^3\gamma + \gamma^3\alpha + \gamma^3\beta)$
$+ (\alpha^2\beta\gamma + \beta^2\gamma\alpha + \gamma^2\alpha\beta)$

$\Rightarrow \Sigma\alpha^2\,\Sigma\alpha\beta = \Sigma\alpha^2\beta + \alpha\beta\gamma(\alpha+\beta+\gamma)$

$\Rightarrow \Sigma\alpha^2\beta = \Sigma\alpha^2\,\Sigma\alpha\beta - \alpha\beta\gamma\,\Sigma\alpha$

$= (p^2-2q)q - pr$

$= p^2q - 2q^2 - pr.$

(v) $\sum \alpha^2\beta\gamma$.

→ We have

$$\sum\alpha^2\beta\gamma = \alpha^2\beta\gamma+\beta^2\gamma\alpha+\gamma^2\alpha\beta$$

$$=\alpha\beta\gamma(\alpha+\beta+\gamma)$$

$$=(-p)(-r) = \underline{pr}.$$

(vi) $\sum\alpha^3$.

→ $\sum\alpha\cdot\sum\alpha^2 = (\alpha+\beta+\gamma)(\alpha^2+\beta^2+\gamma^2)$

$$=\sum\alpha^3 + \sum\alpha^2\beta$$

$$\therefore \sum\alpha^3 = \sum\alpha\cdot\sum\alpha^2 - \sum\alpha^2\beta$$

$$=(-p)(p^2-2q) - (3r-pq) \quad \text{from (i) \& (iii)}$$

$$=-p^3+2pq-3r+pq = 3pq-3r-p^3.$$

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(vii) $\sum\alpha^4$

→ $\because \sum\alpha^4 = \alpha^4+\beta^4+\gamma^4 = (\alpha^2+\beta^2+\gamma^2)^2 - 2(\alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2)$

$$=(p^2-2q)^2 - 2(q^2-2pr)$$

$$=p^4-4p^2q+4q^2-2q^2+4pr$$

$$=p^4-4p^2q+2q^2+4pr.$$

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(viii) $\Sigma\alpha^3\beta^3$.

$$\rightarrow \Sigma\alpha\beta\,\Sigma\alpha^2\beta^2 = (\alpha\beta+\alpha\gamma+\beta\gamma)(\alpha^2\beta^2+\alpha^2\gamma^2+\beta^2\gamma^2)$$

$$= \Sigma\alpha^3\beta^3 + \Sigma\alpha^3\beta^2\gamma$$

$$\Rightarrow \quad = \Sigma\alpha^3\beta^3 + \alpha\beta\gamma\,\Sigma\alpha^2\beta$$

$$\therefore \Sigma\alpha^3\beta^3 = (\Sigma\alpha\beta)(\Sigma\alpha^2\beta^2) - \alpha\beta\gamma\,\Sigma\alpha^2\beta$$

$$= q(q^2-2pr) - (-r)(3r-pq)$$

$$= q^3 - 2pqr + 3r^2 - pqr$$

$$= q^3 - 3pqr + 3r^2.$$

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(ix) $\sum \frac{\beta^2+\gamma^2}{\beta\gamma}$ $\left(\text{i.e } \cancel{\beta}\ \frac{\beta}{\gamma} + \frac{\gamma}{\beta}\right)$

$$\Rightarrow \sum\frac{\beta^2+\gamma^2}{\beta\gamma} = \frac{\beta^2+\gamma^2}{\beta\gamma} + \frac{\gamma^2+\alpha^2}{\gamma\alpha} + \frac{\alpha^2+\beta^2}{\alpha\beta}$$

$$= \frac{\alpha(\beta^2+\gamma^2)+\beta(\gamma^2+\alpha^2)+\gamma(\alpha^2+\beta^2)}{\alpha\beta\gamma}$$

$$= \frac{\Sigma\alpha^2\beta}{\alpha\beta\gamma} = \frac{3r-pq}{(-r)}$$

$$= \frac{pq-3r}{r}$$

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(x) $\sum \frac{\beta^2+\gamma^2}{\beta+\gamma}$

$$\rightarrow \sum \frac{\beta^2+\gamma^2}{\beta+\gamma} = \frac{\beta^2+\gamma^2}{\beta+\gamma} + \frac{\gamma^2+\alpha^2}{\gamma+\alpha} + \frac{\alpha^2+\beta^2}{\alpha+\beta}$$

$$= \frac{\sum (\gamma+\alpha)(\alpha+\beta)(\beta^2+\gamma^2)}{(\beta+\gamma)(\gamma+\alpha)(\alpha+\beta)}$$

Now, ~~nom~~ num

$$= \sum (\beta^2+\gamma^2)(\alpha+\beta)(\alpha+\gamma)$$

$$= \sum (\beta^2+\gamma^2)\{\alpha^2+(\alpha\gamma+\alpha\beta+\beta\gamma)\}$$

$$= \sum (\beta^2+\gamma^2)(\alpha^2+q)$$

$$= \sum \alpha^2(\beta^2+\gamma^2) + q\sum(\beta^2+\gamma^2)$$

$$= 2\sum\alpha^2\beta^2 + 2q\sum\alpha^2$$

$$= 2(q^2-2pr) + 2q(p^2-2q)$$

$$= 2q^2 - 4pr + 2p^2q - 4q^2$$

$$= -2q^2 - 4pr + 2p^2q$$

$$= 2p^2q - 4pr - 2q^2.$$

& Deno. $= (\beta+\gamma)(\gamma+\alpha)(\alpha+\beta)$

$$= (\beta+\gamma)(\gamma\alpha+\beta\gamma+\alpha^2+\alpha\beta)$$

$$= \sum\alpha^2\beta + 2\alpha\beta\gamma$$

$= (3r - pq) + 2(-r)$

$= 3r - pq - 2r = \underline{r - pq}.$

Hence

$$\sum \frac{\beta^2+\gamma^2}{\beta+\gamma} = \frac{2p^2q - 4pr - 2q^2}{r - pq}.$$

Ex:→ If $\alpha, \beta, \gamma, \delta$ be the roots of the biquadratic equation

$x^4 + px^3 + qx^2 + rx + s = 0$ find in the terms of the coefficients, the value of the following symmetric functions:

(i) $\sum \alpha^2$

Solution:→ ∵ $\alpha, \beta, \gamma, \delta$ are the roots of the equation $x^4 + px^3 + qx^2 + rx + s = 0$, we have

$\sum \alpha = -p, \quad \sum \alpha\beta = q, \quad \sum \alpha\beta\gamma = -r, \quad \alpha\beta\gamma\delta = s.$

(i) $\sum \alpha^2 = (\sum \alpha)^2 - 2\sum \alpha\beta$

$= (-p)^2 - 2q \quad = \underline{p^2 - 2q}.$

(ii) $\Sigma \alpha^2 \beta$.

$\rightarrow \Sigma\alpha \Sigma\alpha\beta = (\alpha+\beta+\gamma)(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)$

$= \Sigma\alpha^2\beta + 3\Sigma\alpha\beta\gamma$

$\therefore \Sigma\alpha^2\beta = \Sigma\alpha\Sigma\alpha\beta - 3\Sigma\alpha\beta\gamma$

$= (-p)q - 3(-r) = \underline{3r - pq}.$

(iii) $\Sigma\alpha^2\beta\gamma$.

$\longrightarrow \Sigma\alpha\Sigma\alpha\beta\gamma = (\alpha+\beta+\gamma+\delta)(\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta)$

$= \Sigma\alpha^2\beta\gamma + 4\alpha\beta\gamma\delta.$

$\therefore \Sigma\alpha^2\beta\gamma = \Sigma\alpha\Sigma\alpha\beta\gamma - 4\alpha\beta\gamma\delta$

$= (-p)(-r) - 4s = pr - 4s.$ Ans.

(iv) $\Sigma\alpha^3$

$\rightarrow \Sigma\alpha\Sigma\alpha^2 = (\alpha+\beta+\gamma+\delta)(\alpha^2+\beta^2+\gamma^2+\delta^2)$

$= \Sigma\alpha^3 + \Sigma\alpha^2\beta$

$\therefore \Sigma\alpha^3 = \Sigma\alpha\Sigma\alpha^2 - \Sigma\alpha^2\beta$

$= (-p)(p^2-2q) - (3r-pq)$

$= -p^3 + 2pq - 3r + pq$

$= 3pq - p^3 - 3r.$

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(v) $\Sigma\alpha^3\beta$.

$\rightarrow \Sigma\alpha^3\beta = \alpha^3(\beta+\gamma+\delta) + \beta^3(\gamma+\delta+\alpha) + \gamma^3(\delta+\alpha+\beta)$
$+ \delta^3(\alpha+\beta+\gamma)$

Now. $\Sigma\alpha^2 \Sigma\alpha\beta = (\alpha^2+\beta^2+\gamma^2+\delta^2)(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)$

$= \Sigma\alpha^3\beta + \Sigma\alpha^2\beta\gamma$

$\Rightarrow \Sigma\alpha^3\beta = \Sigma\alpha^2 \cdot \Sigma\alpha\beta - \Sigma\alpha^2\beta\gamma$

$= (p^2-2q)q - (pr-qs)$

$= p^2q - 2q^2 - pr + qs$

(vi) $\Sigma\alpha^4$

$\rightarrow \Sigma\alpha^4 = (\Sigma\alpha^2)^2 - 2\Sigma\alpha^2\beta^2$

$= (p^2-2q)^2 - 2(q^2 - 2pr + 2s)$

$= p^4 - 4p^2q + 4q^2 - 2q^2 + 4pr - 4s$

$= p^4 - 4p^2q + 2q^2 + 4pr - 4s.$

(vii) $\Sigma\alpha^2\beta^2$.

$\rightarrow (\Sigma\alpha\beta)^2 = (\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)^2$

$= \Sigma\alpha^2\beta^2 + 2\Sigma\alpha^2\beta\gamma + 6\alpha\beta\gamma\delta$

$\therefore \Sigma\alpha^2\beta^2 = (\Sigma\alpha\beta)^2 - 2\Sigma\alpha^2\beta\gamma - 6\alpha\beta\gamma\delta$

$= q^2 - 2(pr - 4s)$

$= q^2 - 2pr + 8s - 6s$

$= q^2 - 2pr + \underline{\underline{2s}}$.

(viii) $\Sigma(\alpha-\beta)^2(\gamma-\delta)^2$

$\longrightarrow \Sigma(\alpha^2+\beta^2-2\alpha\beta)(\gamma^2+\delta^2-2\gamma\delta)$

$= \cancel{8}\ 2\Sigma\alpha^2\beta^2 - 2\Sigma\alpha^2\beta\gamma + 12\alpha\beta\gamma\delta$

$= 2(q^2 - 2pr + 2s) - 2(pr - 4s) + 12s$

$= 2q^2 - 4pr + 4s - 2pr + 8s + 12s$

$= 2q^2 - 6pr + 24s$.

(ix) $\Sigma(\gamma\alpha + \beta\delta)(\alpha\beta + \gamma\delta) = \Sigma\alpha^2\beta\gamma$

$= pr - 4s$.

Ans.